Superhedging and distributionally robust optimization with neural networks

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joint work with Michael Kupper and Mathias Pohl

Robust Techniques in Quantitative Finance
University of Oxford

Robust optimization problems

Setup:

- Set of probability measures $\mathcal{Q}$ on $\mathbb{R}^d$
- Continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$.

Objective:

- Solve

$$(P) := \sup_{\nu \in \mathcal{Q}} \int f \, d\nu$$

Different choices of $\mathcal{Q}$ lead to

- Constrained optimal transport (see e.g. Ekren and Soner (2018), includes martingale optimal transport)
- Distributionally robust optimization (see e.g. Esfahani and Kuhn (2015), Blanchet and Murthy (2016), Bartl et al. (2017), Obłój and Wiesel (2018))
- Mixtures of the above (Gao and Kleywegt (2017))

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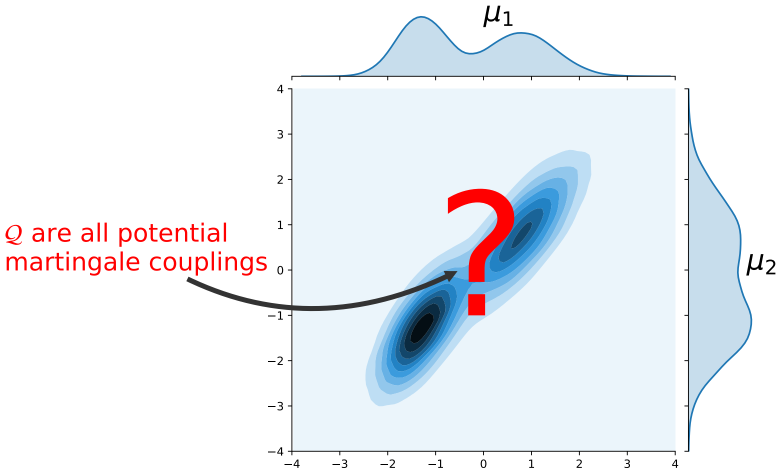
Example: Martingale optimal transport (MOT)

Stock S_t for times $t = 1, 2$ has known marginals $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R})$.

$\mathcal{Q} = \{\nu \in \mathcal{P}(\mathbb{R}^2) : \nu_1 = \mu_1, \nu_2 = \mu_2, \text{ if } (S_1, S_2) \sim \nu, \text{ then } \mathbb{E}[S_2 | S_1] = S_1\}$

The function f can model an exotic option, e.g. $f(s_1, s_2) = (s_2 - K s_1)^+$.



Numerical approach: Discretization

Idea: Reduce problem (P) to a finite dimensional problem (P_n) by going over to a discrete space.

In the prior example for instance:

- Approximate marginals

$$\mu_1 \approx \mu_1^n = \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \qquad \mu_2 \approx \mu_2^n = \sum_{i=1}^{n} \beta_i \delta_{y_i}$$

- Replace $\mathcal{Q}$ by

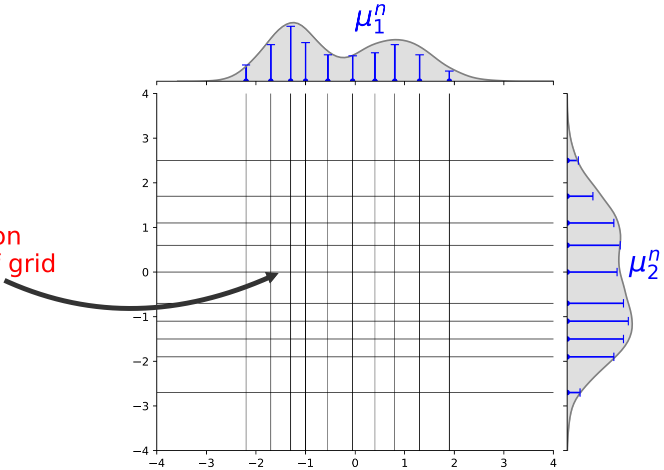
$$\mathcal{Q}_n = \{\nu \in \mathcal{P}(\mathbb{R}^2) : \nu_1 = \mu_1^n, \nu_2 = \mu_2^n, \text{ if } (S_1, S_2) \sim \nu, \text{ then } \mathbb{E}[S_2 | S_1] = S_1\}$$

- Solve $(P_n) = \inf_{\nu \in \mathcal{Q}_n} \int f \, d\nu$ instead of (P).

Numerical approach: Discretization

In the prior example for instance:

Measures in $\mathcal{Q}_n$ only put mass on intersections of grid



Numerical approach: Discretization

Discretization of (P), as well as solving the discrete versions of (P) are studied and applied successfully in a lot of works:

- MOT: Alfonsi et al. (2017), Guo and Obłój (2018)
- OT: See e.g. Peyre and Cuturi (2018)
- Distributionally robust optimization: Esfahani and Kuhn (2015)

Difficulty: Discretization scales badly with dimension.

- In the prior example, (P_n) has $\boldsymbol{n}^2$ parameters.
- In a MOT problem with T time steps, and K dimensional assets, (P_n) has $\boldsymbol{n}^{T \cdot K}$ parameters!

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Alternative: Parametrization

Idea: Work with $\{\nu_\lambda : \lambda \in \Lambda\} \subset \mathcal{Q}$ where Λ is a finite-dimensional parameter space that one can scale independently of dimension.

Problem: No *expressive* sets $\{\nu_\lambda : \lambda \in \Lambda\}$ available that are also *numerically feasible* to work with.

Using the dual formulation

We consider problems $(P) = \sup_{\nu \in \mathcal{Q}} \int f d\nu$ which allow for a dual formulation of the form

$$(D) = \inf_{\substack{h \in \mathcal{H}: \\ h \geq f}} \varphi(h)$$

where $\mathcal{H} \subset C(\mathbb{R}^d)$ and $\varphi : \mathcal{H} \to \mathbb{R}$ is a linear functional.

Example: For the MOT problem from before

- $\mathcal{H} = \{h(x, y) = h_1(x) + h_2(y) + h_3(x) \cdot (y - x) : h_i \in C_b(\mathbb{R})\}$
- $\varphi(h) = \int_{\mathbb{R}} h_1 d\mu_1 + \int_{\mathbb{R}} h_2 d\mu_2$

$\to$ Parametrizing $\mathcal{H}$ is simpler than parametrizing $\mathcal{Q}$! (See also Henry-Labordère (2013))

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Why use neural networks?

(Feed-forward) neural networks are parametrized functions that build on concatenation of several layers of simple functions.

$$\mathbb{R}^k \ni x \mapsto A_l \circ \underbrace{\sigma \circ A_{l-1}}_{(l-1).\text{ layer}} \circ \ldots \circ \underbrace{\sigma \circ A_0}_{1.\text{ layer}}(x)$$

where A_i are affine transformations and $\sigma : \mathbb{R} \to \mathbb{R}$ is a non-linear *activation* function that is applied element-wise.

- Neural networks can approximate a lot of functions with a high, but numerically feasible amount of parameters.
- Successful in practice, even though theoretical understanding of numerical schemes is lacking.

Solution approach using neural networks

$$(D) = \inf_{\substack{h \in \mathcal{H}: \\ h \geq f}} \varphi(h)$$

Step 1: We replace $\mathcal{H}$ by a set of neural network functions $\mathcal{H}^m$. Thereby, trading strategies are then restricted to feed-forward neural networks.

Resulting finite-dimensional problem

$$(D_m) = \inf_{\substack{h \in \mathcal{H}^m: \\ h \geq f}} \varphi(h)$$

Problem: Constraint $h \geq f$ prevents application of numerical schemes based on gradient-descent.

Solution approach using neural networks

$$(D^m) = \inf_{\substack{h \in \mathcal{H}^m: \\ h \geq f}} \varphi(h)$$

Step 2: Penalize the inequality constraint $h \geq f$.

$$(D^m_\theta) = \inf_{h \in \mathcal{H}^m} \varphi(h) + \int \infty \max\{f - h, 0\} d\theta$$

If θ gives positive mass of every open ball, then $(D^m_\theta) = (D^m)$.

Implementation problem: The mapping $x \mapsto \infty \max\{0, x\}$ has no useful gradients.

Solution approach using neural networks

$$(D_\theta^m) = \inf_{h \in \mathcal{H}^m} \varphi(h) + \int \infty \max\{f - h, 0\} d\theta$$

Step 3: Approximate $x \mapsto \infty \max\{x, 0\}$ by a sequence of differentiable nondecreasing convex functions $(\beta_\gamma)_{\gamma > 0}$, e.g. $\beta_\gamma = \gamma \max\{0, x\}^2$.

$$(D_{\theta,\gamma}^m) = \inf_{h \in \mathcal{H}^m} \varphi(h) + \int \beta_\gamma(f - h) d\theta$$

Solution approach: Overview

Label	Statement	Description
(D)	$\inf_{\substack{h\in\mathcal{H}:\\ h\geq f}} \varphi(h)$	initial problem
(D^m)	$\inf_{\substack{h\in\mathcal{H}^m:\\ h\geq f}} \varphi(h)$	finite dimensional version of (D)
(D^m_θ)	$\inf_{h\in\mathcal{H}^m} \varphi(h) + \int \infty(f-h)^+ \, d\theta$	dominated version of (D^m)
$(D^m_{\theta,\gamma})$	$\inf_{h\in\mathcal{H}^m} \varphi(h) + \int \beta_\gamma(f-h)\, d\theta$	penalized version of (D^m_θ)

Table: Summary of problems occurring in the approach.

Solution approach: Theoretical results

Under mild assumptions on $\mathcal{H}$, $\mathcal{H}^m$ and θ it holds

$$(D^m) \to (D) \text{ for } m \to \infty \tag{1}$$

$$(D^m_{\theta,\gamma}) \to (D^m) \text{ for } \gamma \to \infty \tag{2}$$

Related results

- To (1): E.g. Bühler et al. (2018),
- To (2): E.g. Cominetti and San Martín (1994), Cuturi (2013), many others related to *Sinkhorn distance*, *Bregman projection* and *Schrödinger problem*.

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Numerics: MOT example

MOT as in introduction

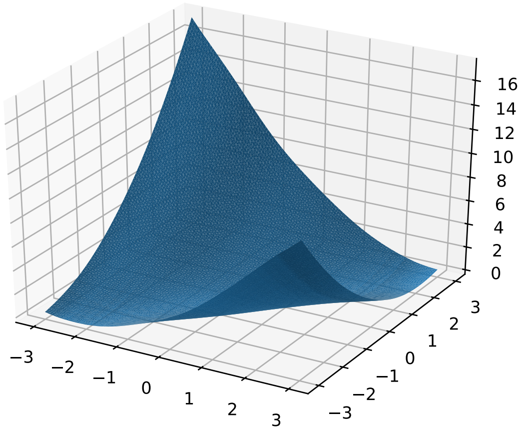
$$f(s1, s2) = (s2 - s1)^+$$

Marginals μ_1, μ_2: Some mixtures of normals

	Neural network	Discretization
Parameter	NN with 4 layers & hidden dimension 64, $\theta = \mu_1 \otimes \mu_2$, $\beta_\gamma(x) = 1000 \max\{0, x\}^2$	$n = 600$ (Relaxed martingale constraint: $\varepsilon = 10^{-6}$)
Optimizer	Approximate dual optimizer $\hat{h}$	Approximate coupling $\hat{\nu}$
Superhedging price	0.2956	0.2990
Subhedging price (inf over $\mathcal{Q}$ instead)	0.0889	0.0844

MOT: Dual optimizer

Superhedging



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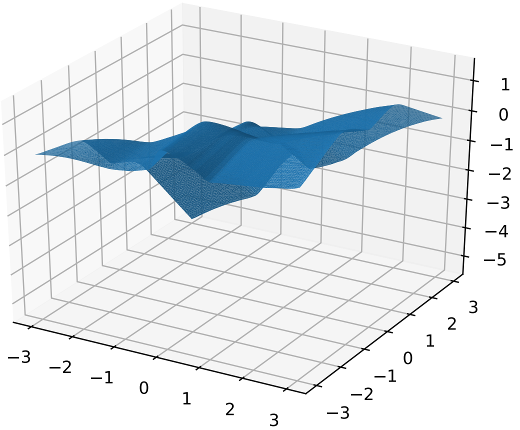

Back to the primal

Goal: Use the neural network solution of the dual to obtain a (near) optimal measure of the primal!

The problem

$$(D_{\theta,\gamma}) = \inf_{h \in \mathcal{H}} \varphi(h) + \int \beta_\gamma(f - h) d\theta$$

has a primal formulation

$$(P_{\theta,\gamma}) = \sup_{\nu \in \mathcal{Q}} \int f d\nu - \int \beta_\gamma^* \left(\frac{d\nu}{d\theta} \right) d\theta$$

and if $(D_{\theta,\gamma})$ has an optimizer $\hat{h}$, one can often show that $\hat{\nu}$ given by

$$\frac{d\hat{\nu}}{d\theta} = \beta_\gamma'(f - \hat{h})$$

is an optimizer of $(P_{\theta,\gamma})$.

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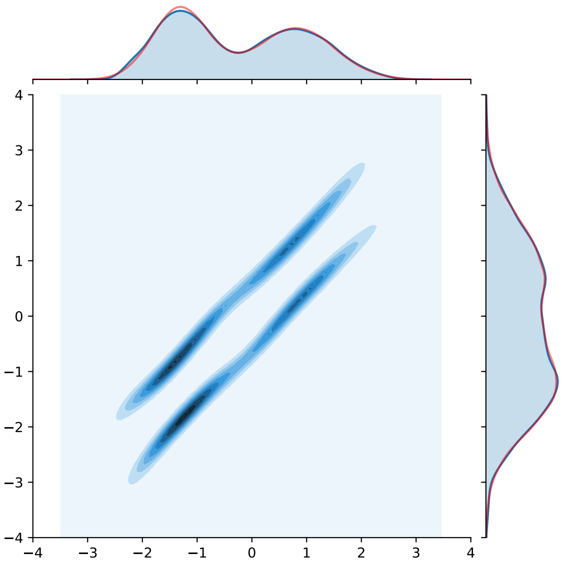
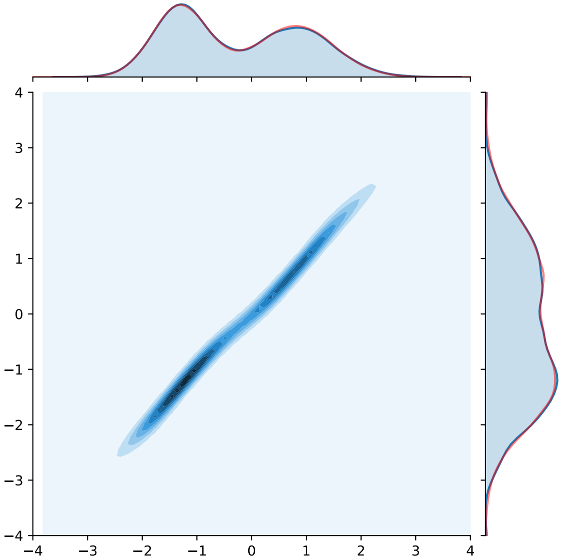
MOT: Primal optimizer

Approximately optimal coupling: Superhedging



MOT: Primal optimizer

Approximately optimal coupling: Subhedging



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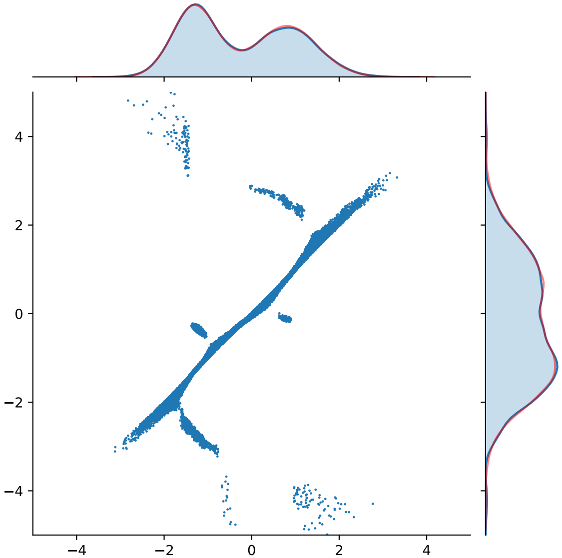
Approximately optimal coupling: Subhedging



Risk aggregation (DNB Case Study - Aas and Puccetti (2014))

- Bank is exposed to 6 types of risks $X_1, ..., X_6$ with known marginal exposures.
- An expert opinion $\bar{\mu} \in \mathcal{P}(\mathbb{R}^6)$ about the joint distribution of $(X_1, ..., X_6)$ is given.

Goal: Calculate bounds on

$$AVaR_\alpha \left(\sum_{i=1}^{6} X_i \right)$$

under constraints that

- $(X_1, ..., X_6)$ have marginals $\mu_1, ..., \mu_6$
- Joint distribution $\nu \sim (X_1, ..., X_6)$ is in a Wasserstein ball around $\bar{\mu}$ of a given radius ρ: $W_p(\bar{\mu}, \nu) \leq \rho$

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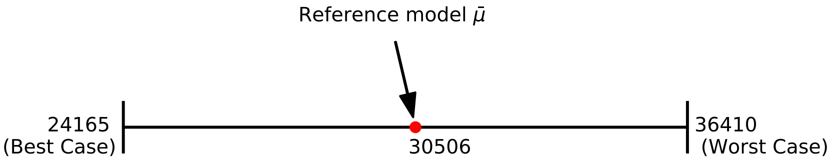
Bank risk aggregation (DNB Case Study)

Using only marginal information:

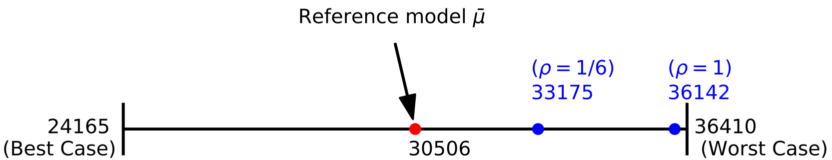
Average Value at Risk for $\alpha = 0.95$



Bank risk aggregation (DNB Case Study)

Worst case around reference model:

Average Value at Risk for $\alpha = 0.95$



Worst case over Wasserstein ball around $\bar{\mu}$ of radius ρ

Thank you

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